# Effective Permittivities for Second-Order Accurate FDTD Equations at Dielectric Interfaces

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***Abstract*—In Yee's finite-difference time-domain (FDTD) scheme, effective permittivities are often used to account for offsets of dielectric interfaces from grid nodes. The specific values of these effective permittivities must be chosen in such a way that the second-order accuracy of the scheme is preserved. It is shown in this letter that, contrary to more elaborate techniques proposed recently for the development of these effective permittivities, a rigorous application of the integral forms of Maxwell's curl equations on the Yee's lattice leads to the desired values in a straightforward fashion. Numerical experiments in a two-dimensional (2-D) cavity are used to verify that the calculated effective permittivities preserve the second-order accuracy of the FDTD scheme.**

***Index Terms*—Convergence of numerical methods, FDTD methods, numerical analysis, permittivity.**

## I. Introduction

SINCE its original introduction by Yee [1] for the numerical simulation of electromagnetic field interactions in homogeneous media, the finite-difference time-domain(FDTD) method has been enhanced significantly and has become one of the most effective methods for handling geometries of high material complexity [2]. One of the issues that continues to receive attention by practitioners in the field is the impact of field discontinuities at material interfaces on the second-order accuracy of the finite-difference approximations of the spatial derivatives on Yee's lattice. For example, for the case of a dielectric material interface, properly constructed averages of the media permittivities have been proposed for accurate updating of the field components in the vicinity of the interface [3], [4]. Typically, the calculation of the effective permittivities is based on a Taylor series analysis of the electromagnetic field quantities at the dielectric interface. Such an analysis must be done carefully due to the discontinuity of the fields and/or its derivatives at dielectric interfaces. Subsequently, the demonstration of the accuracy of the calculated effective permittivities is based on the investigation of a derivative electromagnetic wave quantity, such as the reflection coefficient at the dielectric interface [5], rather than the examination of the second-order accuracy of the resulting FDTD equations.

It is shown in this letter that the calculation of effective permittivities for second-order accurate finite difference schemes at a dielectric interface can be effected in a systematic and straightforward manner through the discretization of the integral forms of Maxwell's equations over finite volumes containing the interface. The proposed analysis confirms the validity of the effective permittivities obtained in [5]. Furthermore, numerical experiments are reported that verify the second-order accuracy of the FDTD scheme that uses the derived effective permittivities for the normal and tangential electric field components at the dielectric interface.

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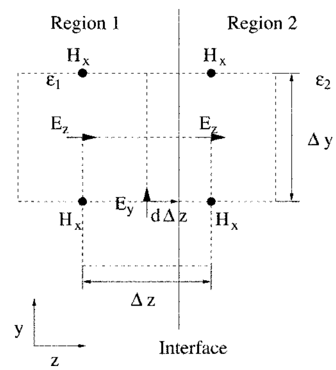


Fig. 1. Placement of electric and magnetic field nodes near a dielectric interface for the case of 2-D TE$_x$ polarization. Dotted lines represent the finite-volume cells used for the development of the discrete equations.

## II. Calculation of Effective Permittivities

Our development is based on the geometry of Fig. 1 where the FDTD approximation of Maxwell's equations for the case of two-dimensional (2-D) TE$_x$ polarization case is considered. The dielectric interface is parallel to the $y$ axis. Without loss of generality, the dielectric media in regions 1 and 2 are assumed to be lossless, with permittivities $\epsilon_1$ and $\epsilon_2$, respectively. We investigate the general case where the dielectric interface is offset from the grid points or nodes. The dielectric interface offset parameter, $d$, is defined as the distance of the interface from the nearest tangential electric field node, $E_y$, normalized to the grid size $\Delta z$. Thus, it is $0 \leq d \leq 0.5$. The temporal approximation of Ampere's law in integral form

$$\frac{\partial}{\partial t} \iint_S \mathbf{D} \cdot \mathbf{ds} \oint_C \mathbf{H} \cdot \mathbf{dl} \tag{1}$$

yields

$$\iint_S \mathbf{D}^{n+1} \cdot \mathbf{ds} = \iint_S \mathbf{D}^{n} \cdot \mathbf{ds} + \Delta t \oint \mathbf{H}^{n+1/2} \cdot \mathbf{dl} + O((\Delta t)^3). \tag{2}$$

Effective permittivities will be derived for both components of the electric field. The effective permittivity, $\epsilon_y^*$, associated with the tangential electric component node $E_y$, is considered first. The discretization of the spatial integrals in (2) related to the temporal update of node $E_y$ (see Fig. 1) yields

$$\begin{aligned}\epsilon_1 E_{y_{j,k}}^{n+1}(\tfrac{1}{2}+d)\Delta z + \epsilon_2 E_{y_{j,k}}^{n+1}(\tfrac{1}{2}-d)\Delta z + O((\Delta z)^3)\\ = \epsilon_1 E_{y_{j,k}}^{n}(\tfrac{1}{2}+d)\Delta z + \epsilon_2 E_{y_{j,k}}^{n}(\tfrac{1}{2}-d)\Delta z + O((\Delta z)^3)\\ + \Delta t\left[H_{x_{j,k+1/2}}^{n+1/2} - H_{x_{j,k-1/2}}^{n+1/2} + O((\Delta z)^2)\right]\\ + O((\Delta t)^3). \qquad (3)\end{aligned}$$

In the above equations, use was made of the fact that $E_{y_{j,k}}$ is continuous across the interface and maintains the second-order spatial accuracy for its piecewise constant representation over each cell. Rearranging (3), we obtain

$$\begin{aligned}E_{y_{j,k}}^{n+1} = E_{y_{j,k}}^{n} + \frac{\Delta t}{\epsilon_y^* \Delta z}\left(H_{x_{j,k+1/2}}^{n+1/2} - H_{x_{j,k-1/2}}^{n+1/2}\right)\\ + O((\Delta t)^2 + (\Delta z)^2) \qquad (4)\end{aligned}$$

where

$$\epsilon_y^* = (\tfrac{1}{2}+d)\epsilon_1 + (\tfrac{1}{2}-d)\epsilon_2. \qquad (5)$$

Thus, (5) is defined as the effective permittivity for the tangential electric field node. In [6], similar discrete integral approach was used to account for thin material sheets.

The effective permittivity, $\epsilon_z^*$ for the normal electric field node $E_z$ in the right side of the $E_y$ node can be derived in a similar manner. Considering the geometry of Fig. 1, the discontinuity of the normal electric field component across the interface leads us to write

$$E_{z_{j,k}} = dE_{z1_{j,k}} + (1-d)E_{z2_{j,k}} \qquad (6)$$

where $E_{z1_{j,k}}$, $E_{z2_{j,k}}$ represent the average values of $E_z$ in the sub-cells in regions 1 and 2, respectively. Then, it is

$$\begin{aligned}E_{z_{j,k}}^{n+1} &= dE_{z1_{j,k}}^{n+1} + (1-d)E_{z2_{j,k}}^{n+1}\\ &= d\left[E_{z1_{j,k}}^{n} - \frac{\Delta t}{\epsilon_1 \Delta y}\left(H_{x_{j+1/2,k}}^{n+1/2} - H_{x_{j-1/2,k}}^{n+1/2}\right) + O((\Delta t)^2 + (\Delta y)^2)\right]\\ &\quad + (1-d)\left[E_{z2_{j,k}}^{n} - \frac{\Delta t}{\epsilon_2 \Delta y}\cdot\left(H_{x_{j+1/2,k}}^{n+1/2} - H_{x_{j-1/2,k}}^{n+1/2}\right) + O((\Delta t)^2 + (\Delta y)^2)\right]\\ &= E_{z_{j,k}}^{n} - \frac{\Delta t}{\epsilon_z^* \Delta y}\left(H_{x_{j+1/2,k}}^{n+1/2} - H_{x_{j-1/2,k}}^{n+1/2}\right)\\ &\quad + O((\Delta t)^2 + (\Delta y)^2) \qquad (7)\end{aligned}$$

where

$$\frac{1}{\epsilon_z^*} = \frac{d}{\epsilon_1} + \frac{1-d}{\epsilon_2}. \qquad (8)$$

In [5], (5) and (8) were derived using the accuracy of the reflection coefficient at the dielectric boundary as a metric. For the case of $-0.5 < d < 0$ considered in [5] it is easily verified that their result for the effective permittivity for the tangential electric field is in agreement with our result (5). However, because of the negative value of $d$, the effective permittivity for the normal component must be assigned to the $E_z$ node to the left of the $E_y$ node. Taking this into account, it is easily verified that the analysis performed above for the derivation of $\epsilon_z^*$ leads to

$$\frac{1}{\epsilon_z^*} = \frac{1+d}{\epsilon_1} - \frac{d}{\epsilon_2}, \qquad \text{for } -0.5 < d < 0 \qquad (9)$$

in agreement with the result in [5].

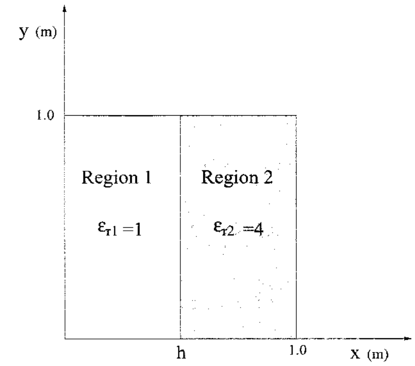


Fig. 2. Geometry of a 2-D cavity with perfectly conducting walls. Region 1 is free space with relative permittivity $\epsilon_r = 1$. Region 2 is filled with a lossless dielectric of $\epsilon_r = 4$.

## III. Numerical Experiments

In order to examine whether the derived effective permittivities lead to an FDTD scheme that exhibits second-order convergence, the electromagnetic response of the two-dimensional cavity shown in Fig. 2 is analyzed. All walls are assumed to be perfect electric conductors. Thus, analytic solutions for the $TE_x$ modes are readily available. They constitute the reference results for the calculation of the $L_2$ norm of the numerical solution that is used to check the convergence rate of the FDTD scheme. For a given time $t = n\Delta t$, the $L_2$ norm of the error in the computed field data $E_{z_{j,k}}^{n}$ is

$$L_2(t) = \sqrt{\frac{\sum_{j=1}^{j=M}\sum_{k=1}^{k=N}\left(E_{z_{j,k}}^{n} - E_{z exact}\right)^2}{M \times N}} \quad \text{for } 0 \le t \le t_{\max} \qquad (10)$$

where $y = (j-1)\Delta y$, $z = (k-1)\Delta z$, $E_{z exact}(y, z, t)$ is the exact solution, and $M$, $N$ are the number of nodes in the $y$ and $z$ directions, respectively. For all calculations in this letter, $\Delta y = \Delta z$. In addition, $t_{\max} = 26$ ns , which corresponds to about 10 periods of the resonance frequency, $f_r$, of the cavity. Convergence rate is monitored by examining the maximum value of the $L_2$ norm of the error, $\max(L_2(t))$ with cell size and time step as parameters.

The case where the material interface coincides with the tangential electric field nodes at $z = 0.5$ m is considered first. This corresponds to the case of $d = 0$ in (5) and (8). For this case,

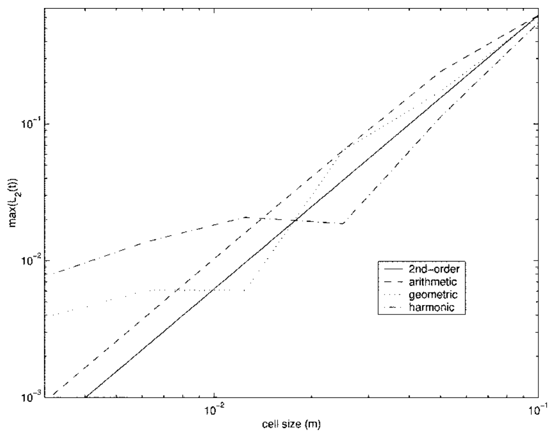


Fig. 3. Comparison of $L_2$ norm errors for three different choices for the effective permittivity for the case $h = 0.5$ m and $d = 0$. The mode under study is the TE$_{x23}$ mode with resonant frequency $f_r = 374.92$ MHz. The solid line indicates the reference second-order convergence slope.

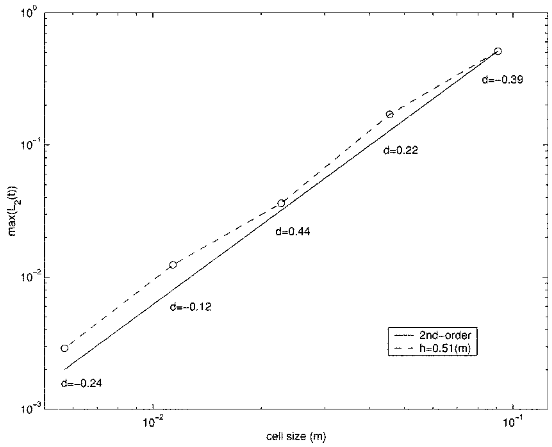


Fig. 4. $L_2$ norm errors as a function of the cell size for the cavity of Fig. 2 with $h = 0.51$ m. The mode under study is the TE$_{x23}$ mode with resonant frequency $f_r = 377.49$ MHz. The solid line indicates the reference second-order convergence slope.

several different values for the effective permittivity for the update of the tangential electric field at the interface have been proposed in the literature [3], [7], [8]. They are based on mean values of the permittivities of the two regions

$$\epsilon_y^* = \begin{cases} \dfrac{\epsilon_1 + \epsilon_2}{2}, & \text{(arithmetic mean)} \\ \dfrac{2\epsilon_1\epsilon_2}{\epsilon_1 + \epsilon_2}, & \text{(harmonic mean)} \\ \sqrt{\epsilon_1\epsilon_2}, & \text{(geometric mean).} \end{cases} \tag{11}$$

Fig. 3 presents the convergence slopes for these three effective permittivities. The solid line indicates the second-order convergence reference slope. The geometric and harmonic means result in better accuracy for cell sizes in the range of $0.05\lambda_0$—$0.1\lambda_0$, where $\lambda_0$ is the free-space wavelength. However, both of them fail to maintain second-order convergence as the cell size decreases. On the other hand, while the arithmetic mean has inferior accuracy for coarser grids, it exhibits second order accuracy regardless of the grid resolution and outperforms the other two means for the case of very fine grids.

Finally, the convergence of the FDTD scheme based on the use of the effective permittivities proposed in this letter for the case of an offset dielectric interface from the FDTD lattice is examined. For this purpose, the dielectric interface in the geometry of Fig. 2 is set at $h = 0.51$ m. This causes the offset parameter $d$ to assume values in an arbitrary fashion in the range $-0.5 < d \leq 0.5$ as the cell size is varied. In Fig. 4, the plot of the maximum value of the $L_2$ norm of the error in the numerical solution as the grid size is reduced demonstrates clearly the second-order convergence of the FDTD scheme. As before, the convergence slope of the FDTD solution parallels the solid line indicating the reference second-order convergence slope, independent of the value of $d$. We conclude that use of (5) and (8) enables the application of the standard FDTD method for electromagnetic simulation of geometries with dielectric interfaces offset from the lattice without compromising the second-order accuracy of the scheme.

## IV. CONCLUSIONS

In conclusion, it is shown in this letter that the effective permittivities required for modeling with second order accuracy dielectric interfaces in the standard FDTD method can be calculated in a systematic manner from the discrete approximation of integral from of Maxwell's curl equations over finite volumes containing the interface. Through numerical experiments it was demonstrated that the second-order accuracy of the discrete solution is achieved irrespective of the offset of the interface from the lattice.

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